Differential Equations:-

\# Def^n: An equation involving the dependent variable, an independent variable & the diff^n. coefficients of various orders is called an. Differential equation (D.E).

Any equation which contains a derivative is called a diff^n equation.

e.g. $\frac{dy}{dx} = \cos x$, $\frac{d^2y}{dx^2} = 0$.

Types:-

(1) Ordinary differential equation:-

Differential equations involving only one independent variable & one or more dependent variables and their derivatives w.r.t. independent variables are called ordinary diff^n eq^n. (O.D.E)

(2) partial differential Equations:-

Differential equations involving two or more independent variables and one or more dependent variables and their partial derivatives w.r.t independent variables are called partial diff^n equations (P.D.E.).

\# Order of a Differential Equation:-

The order of a differential equation is the order of the highest derivative that appears in the equation.

\# Degree of Diff^n equation:-

The degree of a diff^n equation is the degree of highest order differential coefficient or derivative, when the diffⁿ coefficients are free from radicals and fractions.

e.g.

(1) $\frac{dy}{dx} = \frac{x+2y+3}{x-y+1}$, order = 1, degree = 1.

(2) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$, order = 2, degree = 2

Squaring on both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

Solution of a differential Equation:-

A solution or primitive of a diffⁿ equation is any relation between dependent and independent variables which is free from derivatives and which satisfies the differential equation.

General solution of a differential Equation:-

A relation between the dependent & independent variables, which is free from derivatives, which satisfies a given diffⁿ eqⁿ & which contains arbitrary consts equal to order of the diffⁿ equation is called the general solⁿ. (G.S). or complete integral.

Hence, the general solⁿ of a diffⁿ eqⁿ of order n must involve n arbitrary consts.

particular sol^n of a $diff^n$ equation:-

The solution obtained by assigning particular values to the arbitrary constants in G.S. of a $diff^n$ eq^n is called a particular solution or particular integral.

Note:- Total number of arbitrary consts in the general solution is equal to the order of the equation.

Formation of ordinary differential Equations:-

An ordinary differential equation can be formed by eliminating number of arbitrary consts in its general sol^n.

If there is one arbitrary const, the relation or sol^n to be differentiate once with respect to independent variable. and also if there are 'n' arbitrary const. $diff^n$ 'n' times.

E.g. ① Find the D.E. whose G.S is $y = c_1 x + c_2 e^x$

Sol^n:- Consider, $y = c_1 x + c_2 e^x$ ——(1)

As there are two arbitrary const. $diff^n$ twice we get,

$$\frac{dy}{dx} = c_1 + c_2 e^x \quad \text{——(2)}$$

$$\frac{d^2y}{dx^2} = c_2 e^x \quad \text{——(3)}$$

$\therefore$ from eq^n (2) & (3).

$$\frac{dy}{dx} = c_1 + \frac{d^2y}{dx^2}$$

$$\therefore c_1 = \frac{dy}{dx} - \frac{d^2y}{dx^2}$$

substitute in eq^n ①

$$\therefore y = \left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right)x + \frac{d^2y}{dx^2}.$$

i.e. $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is the required diff^n ~~d~~equation.

(2) Form the differential equation of which the primitive is $y = a\sin t + b\cos t$.

Soln: $\therefore$ There are two arbitrary const.

$\therefore$ Diff^n twice w.r.t. 't' we get.

$$\frac{dy}{dt} = a\cos t - b\sin t$$

$$\frac{d^2y}{dt^2} = -a\sin t - b\cos t = -y.$$

$\therefore \frac{d^2y}{dt^2} + y = 0$ is required D.E.

Ordinary Differential Equations of the First order And First degree:-

- An ordinary diff^n equation of the 1st order and 1st degree is of the form

$$M + N\frac{dy}{dx} = 0 \Rightarrow Mdx + Ndy = 0,$$

where M and N are functions of x & y or const.

- The general soln of such equation will contain only one arbitrary const.
- The method of solving a diff^n equation of first order and 1st degree depends upon the type to which it belongs.

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Type (I). Variables separable form (v.s Form)

Many first order differential equations can be reduced to the form
$\frac{dy}{dx} = \frac{f(x)}{g(y)}$ or $\left(\frac{dy}{dx} = \frac{g(y)}{f(x)}\right)$ by algebraic manipulations. Here we find it convenient to write $g(y)\,dy = f(x)\,dx$

$$\Rightarrow \text{or} \left(\frac{dy}{g(y)} = \frac{dx}{f(x)}\right)$$

such an equation is in variable separable form (v.s. form). because the variables x and y are separated so that x appears only on one side of the equation & y appears only on the other side.

The solution is obtained by integrating on both sides giving.

$$\int g(y)\,dy = \int f(x)\,dx + c$$

$$\text{or} \int \frac{dy}{g(y)} = \int \frac{dx}{f(x)} + c$$

eg (1) solve $\frac{dy}{dx} = \frac{x\sin x}{2e^y\sinh y}$

solⁿ:- Given equation is of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

hence it can be expressed in the variable separable form as $g(y)\,dy = f(x)\,dx$.

$$\therefore \frac{dy}{dx} = \frac{x\sin x}{2e^y \sinh y} \Rightarrow 2e^y \sinh y\, dy = x\sin x\, dx$$

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$\therefore$ G.S $\Rightarrow 2\int e^{y} \cdot \sinh y \, dy = \int x \cdot \sin x \, dx + c$

$\Rightarrow \cancel{2}\int e^{y}\frac{(e^{y}-e^{-y})}{\cancel{2}}dy = x(-\cos x) - (1)(-\sin x) + c$

$\Rightarrow \frac{e^{2y}}{2} - y = -x\cos x + \sin x + c$

$\Rightarrow \frac{e^{2y}}{2} - y + x\cos x - \sin x = 0$ is the General soln

(2) solve :- $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

soln:- Given $\Rightarrow \frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

$\Rightarrow \int \frac{dy}{1+y^2} + \int \frac{dx}{1+x^2} = 0$ ..... V.S form.

$\therefore$ G.S $= \tan^{-1}y + \tan^{-1}x = \tan^{-1}c$.

$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}c$

$\therefore \boxed{\frac{x+y}{1-xy} = c}$ is the General soln.

Type-2 :- Differential Equations Reducible to v.s form by using substitution.

(1) Linear substitution:- If the given differential equation is of the form $\frac{dy}{dx} = F(ax+by+c)$ then

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we use the substitution $ax+by+c=u$. Given diff^n equation reduces to V.S. form in the variables u, x

(2) Quotient substitution:-

(i) Also diff^n equation of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ reduces to V.S. form by using substitution

$\frac{y}{x} = u$, $\Rightarrow y = ux$.

$\Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

(ii) Also diff^n eq^n of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$ reduces to V.S. form by using substitution.

$\frac{x}{y} = u$ $\Rightarrow x = uy$.

$\Rightarrow \frac{dx}{dy} = u + y + \frac{du}{dy}$

Eg. (1) solve $\frac{dy}{dx} = 1 - x\tan(x-y)$

Sol^n:- put $x-y=u$.

$\therefore 1 - \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{du}{dx} = x\tan u$.

$\therefore \frac{du}{\tan u} = x\,dx$. ..... V.S. form

$\therefore$ G.S $\Rightarrow$ Integrating, $\int x\,dx - \int \cot u\,du = 0$

$\Rightarrow x^2/2 - \log\sin u = c$

$$\boxed{\Rightarrow \frac{x^2}{2} - \log\sin(x-y) = c}$$

Que(2) solve $x^4 \frac{dy}{dx} + x^3 y - \sec(xy) = 0$. ... (1)

Sol^n. put $xy = u$

$\therefore x\frac{dy}{dx} + y = \frac{du}{dx}$

(1) $\Rightarrow \therefore x^3\left(x\frac{dy}{dx} + y\right) - \sec(xy) = 0$

$x^3 \cdot \frac{du}{dx} - \sec u = 0$

$x^3 \frac{du}{dx} = \sec u$.

$\Rightarrow \cos u\, du = \frac{dx}{x^3}$ ... V.S. form

$\therefore$ G.S $\Rightarrow \int \cos u\, du = \int \frac{dx}{x^3}$

$\Rightarrow \sin u + \frac{1}{2x^2} = c$

$\Rightarrow \sin(xy) + \frac{1}{2x^2} = c$ is the required G.S.

Que(3) solve $y e^{x/y} dx = (x e^{x/y} + y^2) dy$.

Sol^n. $y e^{x/y} dx - x e^{x/y} = y^2$

$\Rightarrow e^{x/y}\left(\frac{dx}{dy} - \frac{x}{y}\right) = y$

put $x = yv$. $\Rightarrow \frac{dx}{dy} = y\frac{dv}{dy} + v$.

$\therefore e^v\left(y\frac{dv}{dy} + v - v\right) = y \Rightarrow e^v \cdot y\frac{dv}{dy} = y \Rightarrow e^v dv = dy$ .. VS form

$\therefore$ G.S $\Rightarrow \int e^v dv = \int dy$

$\Rightarrow e^v - y = c$

$\Rightarrow \boxed{e^{x/y} - y = c}$ is the required G.S.

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Type (III) Homogeneous differential Equation:-

(A) Homogeneous functions:- A function $f(x,y)$ is said to be homogeneous if degree of each term of f is same.

e.g. $f(x,y) = x^2+xy+y^2$ or $\sqrt{x+y}$

(B) Homogeneous differential equation:-

A differential equation $Mdx+Ndy=0$ or $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ is said to be homogeneous if M and N are homogeneous functions in x and y of same degree.

These equations are reducible to V.S form by changing the dependent variable from y to u by the substitution

$$y = ux$$

$$\therefore \frac{dy}{dx} = u + x\frac{du}{dx}.$$

Examples:-

① solve $x\frac{dy}{dx} + \frac{y^2}{x} = y$.

Sol^n:- Given differential equation is homogeneous.

$$\therefore \frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x} \quad \text{——} (1)$$

put $\frac{y}{x} = u \Rightarrow y = ux$

$$\therefore \frac{dy}{dx} = u + x\frac{du}{dx}$$

$\therefore$ ~~G.S~~ $\Rightarrow$ from eq^n (1),

$$x\frac{du}{dx} + u + u^2 = u$$

$$\Rightarrow x\,{}^{du}/_{dx} + u^2 = 0$$

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$$G.S \Rightarrow \int \frac{du}{u^2} + \int \frac{dx}{x} = 0$$

$$\therefore \frac{-1}{u} + \log x = c$$

$$\therefore \boxed{\log x - \frac{x}{y} = c}$$ which is the required General soln.

(2) Solve:- $(y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0$

$\Rightarrow$ As M and N are homogeneous functions in x and y of degree 4

$\therefore$ Given D.E is homo. It can be written as

$$\boxed{\frac{dy}{dx}} = \frac{2x^3y - y^4}{x^4 - 2xy^3} \quad \text{--- (1)}$$

put $y = ux \Rightarrow \frac{dy}{dx} = u + x \cdot \frac{du}{dx}$.

substituting values of y and $\frac{dy}{dx}$ in eqn (1) we get,

$$u + x\frac{du}{dx} = \frac{2x^4u - x^4u^4}{x^4 - 2x^4u^3}$$

$$\therefore x\frac{du}{dx} = \frac{2u - u^4}{1 - 2u^3} - u$$

$$x\frac{du}{dx} = \frac{u + u^4}{1 - 2u^3}$$

$$\therefore \frac{1 - 2u^3}{u + u^4} du = \frac{dx}{x} \quad \text{(V.S.F)}$$

$$\Rightarrow \frac{dx}{x} = \left(\frac{1}{u} - \frac{3u^2}{1 + u^3}\right) du$$

Integrating,

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$$\therefore \int \frac{dx}{x} = \int \frac{1}{u}du - \int \frac{3u^2\,du}{1+u^3}$$

$\Rightarrow \log x = \log u - \log(1+u^3) + \log c$

$$\Rightarrow \frac{x(1+u^3)}{u} = c$$

$\Rightarrow x^3 + y^3 = cxy$ is the general solⁿ.

Type-(IV):- Non-Homogeneous differential equations Reducible to Homogeneous form

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$ is called non-Homo. differential equation. — (1)

case (1): If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.

In this case, the expressions $a_1x + b_1y$ & $a_2x + b_2y$ will always have a common factor of the form $lx + my$.

we put $lx + my = u$, then eqⁿ (1) ~~become~~ reduced to v.s. form in the variables u, x.

case (2):- If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

In this case to reduce equation (1) to homogeneous form, we substitute.

$x = X + h$, $y = Y + K$.

where, h & K are constants to be determin

Also. $dx = dX$, $dy = dY$

$\therefore \frac{dy}{dx} = \frac{dY}{dX}$ $\therefore$ eqⁿ (1) becomes.

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$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

choose h and k such that equation will become homogeneous in X & Y.

i.e. $a_1h + b_1k + c_1 = 0$ & $a_2h + b_2k + c_2 = 0$

we get,

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous equation in X & Y.

put $Y = VX$

$\Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$ then eq^n reduces to.

V.S. form

Examples:-

1. solve:- $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$.

Soln:- The given equation is.

$$\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5} = \frac{(x-y)+3}{2(x-y)+5} \quad \text{---(1)}.$$

put $x - y = u$

$$1 - \frac{dy}{dx} = \frac{du}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{du}{dx}.$$

$\therefore$ from eq^n (1), $1 - \frac{du}{dx} = \frac{u+3}{2u+5}$

$$\therefore -\frac{du}{dx} = \frac{u+3}{2u+5} - 1$$

$$\Rightarrow \frac{du}{dx} = \frac{u+2}{2u+5}.$$

$$\therefore \frac{2u+5}{u+2}\,du = dx$$

$$\therefore \frac{2(u+2)+1}{u+2}\,du = dx$$

$$\therefore \left(2+\frac{1}{u+2}\right)du = dx$$

Integrating, we get

$$2u + \log(u+2) = x + c$$

$$\Rightarrow 2(x-y) + \log(x-y+2) = x + c.$$

which is the required General sol^n.

(2) solve $(2x+y-3)\,dy = (x+2y-3)\,dx$.

Sol^{no}:-

The Given eq^n is,

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \quad \therefore \text{As } \frac{1}{2} \neq \frac{2}{1} \quad \text{---} (1)$$

put $x = X+h$, $y = Y+K$ $\Rightarrow dx = dX$ & $dy = dY$.

Substituting in eq^n (1).

$$\therefore \frac{dY}{dX} = \frac{(X+2Y)+(h+2K-3)}{(2X+Y)+(2h+K-3)} \quad \text{---} (2)$$

choose h, K such that $h+2K-3=0$
& $2h+K-3=0$

$\Rightarrow h = K = 1$

$\therefore$ from eq^n (2), we get

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

which is homogeneous.

$\therefore$ put $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

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$$V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+2V}{2+V} - V$$

$$\therefore X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

$$\Rightarrow \frac{2+V}{1-V^2}dV = \frac{dX}{X}$$

$$\Rightarrow \left(\frac{1/2}{1+V} + \frac{3/2}{1-V}\right)dV = \frac{dX}{X}$$

Integrating,

$$\frac{1}{2}\int\frac{dV}{1+V} + \frac{3}{2}\int\frac{dV}{1-V} = \int\frac{dX}{X}$$

$$\log(1+V) - 3\log(1-V) = 2\log X + c$$

$$\Rightarrow \log\frac{(1+V)}{(1-V)^3X^2} = \log c_1$$

$$\Rightarrow \frac{1+V}{(1-V)^3X^2} = c_1$$

$$\Rightarrow \frac{1+Y/X}{\left(1-\frac{Y}{X}\right)^3X^2} = c_1$$

$$\Rightarrow \frac{X+Y}{(X-Y)^3} = c_1$$

$$\Rightarrow X+Y = c_1(X-Y)^3 \quad \text{———} (3)$$

Now, $X = x-1$, $Y = y-1$

$\therefore (3) \Rightarrow x+y-2 = c_1(x-y)^3$ is the G. S.

Type(5):- Exact Differential equation:-

Consider a differential equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$.

If there exists a function $u(x,y)$ such that $Mdx + Ndy = du$ then the differential equation is called as an exact differential equation.

Condition of Exactness:-

The necessary and sufficient condition that $Mdx + Ndy = 0$ be exact is,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

when the condition of exactness is satisfied the general solution can be obtained by the following rules.

Rule 1:- $$\boxed{\int M dx + \int \left(\text{Terms of } N \text{ not containing } x\right) dy = c}$$

i.e. integrate Mdx w.r.t. x treating y const. integrate only those terms in Ndy which are free from x w.r.t y and equate their sum to a constant.

Rule 2:- If N has no term which is free from x then $\int_{y=\text{const}} M dx = c$ is the G.S.

Rule 3:- Sometimes we may write the G.S. by using following rule.

$$\int_{x\text{-const}} N dy = \int \left(\text{Terms of } M \text{ not containing } y\right) dx = c$$

Remark:- Sometimes an equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ becomes exact if $\boxed{b_1 = -a_2}$,

because the equation can be written as,

$(a_1x+b_1y+c_1)dx - (a_2x+b_2y+c_2)dy = 0$ with $\frac{\partial M}{\partial y} = b_1$ & $\frac{\partial N}{\partial x} = -a_2$ accordingly, the soln is given by, $\int (a_1x+b_1y+c_1)dx - \int (b_2y+c_2)dy = c$

(treat y = const)

Examples:-

(1) Solve $(x+y-2)dx + (x-y+4)dy = 0$

Soln:- Given eqn is of the type $Mdx + Ndy = 0$

Here $M = x+y-2$ & $N = x-y+4$

$\therefore \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$

$\therefore$ Given eqn is exact differential equation

Its G.S. is given by

$$G.S \Rightarrow \int_{y\text{-const.}} (x+y-2)dx + \int_{\text{No } x \text{ terms}} (-y+4)dy = c$$

$$\Rightarrow \frac{x^2}{2} + xy - 2x - \frac{y^2}{2} + 4y = c$$

$$\Rightarrow x^2 - y^2 + 2xy - 4x + 8y = c_1$$ which is required General soln.

(2) Solve $\frac{dy}{dx} = \frac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$.

Soln:- Given diffn eqn can be written as.

$$(\tan y - 2xy - y)dx + (x\tan^2 y - x^2 - \sec^2 y)dy = 0 \quad \text{(1)}$$

$\therefore$ Here $M = \tan y - 2xy - y$

& $N = x\tan^2 y \cancel{- \sec^2 y} - x^2 - \sec^2 y$.

~~Type~~ :- $\therefore \frac{\partial M}{\partial y} = \sec^2 y - 2x - 1 = \tan^2 y - 2x$

& $\frac{\partial N}{\partial x} = \tan^2 y - 2x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ eqn ① is exact.

$\therefore$ G.S $\Rightarrow \underbrace{\tan y \int dx - 2y\int x - y\int dx}_{y\text{-const}} - \underbrace{\int \sec^2 y\, dy}_{\text{No } x \text{ terms}} = c$

$\Rightarrow x\tan y - x^2 y - xy - \tan y = c$ is G.s.

(3) solve. $(y^2 \cdot e^{xy^2} + 4x^3)dx + (2xy\, e^{xy^2} - 3y^2)dy = 0$

Soln:- Here,

$M = y^2 \cdot e^{xy^2} + 4x^3$, $N = 2xy\, e^{xy^2} - 3y^2$

$\therefore \frac{\partial M}{\partial y} = 2y \cdot e^{xy^2} + y^2 e^{xy^2}(2xy)$

& $\frac{\partial N}{\partial x} = 2y e^{xy^2} + 2xy\, e^{xy^2}(y^2)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Given diffn eqn is exact

$\Rightarrow$ G.s is $\Rightarrow y^2 \int e^{xy^2} dx + 4\int x^3 dx - 3\int y^2 dy = c$

$\frac{\cancel{y^2}\, e^{xy^2}}{\cancel{y^2}} + x^4 - y^3 = c$

$\therefore e^{xy^2} + x^4 - y^3 = c$ is the required G.s.

Type 6:- Equations Reducible to exact form by using Integrating factor:-

Integrating factor:- A function $f(x,y)$ is said to be an Integrating factor (I F) of the equation $Mdx + Ndy = 0$. If it is possible to obtain a function $u(x,y)$ such that

$F(Mdx + Ndy) = du$.

In other words, an I.F. is a multiplying factor by which the equation can be made exact.

Rules for finding Integrating factor of the equation $Mdx + Ndy = 0$ when it is not exact.

Rule 1:- If $x \cdot M + y \cdot N \neq 0$ & the given diffⁿ eqⁿ is homogeneous. then

$$I \cdot F = \frac{1}{x \cdot M + y \cdot N}.$$

Rule 2:- If $x \cdot M - y \cdot N \neq 0$ & the given D.E has the form $y \cdot f_1(xy)dx + x \cdot f_2(xy)dy = 0$ then,

$$I \cdot F = \frac{1}{xM - y \cdot N}.$$

Rule 3:- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (say) then

$$I \cdot F = e^{\int f(x)dx}.$$

Rule 4:- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \phi(y)$ (say) then

$$I \cdot F = e^{\int \phi(y)\, dy}$$

Rule 5:- If the equation $Mdx + Ndy = 0$ can be written as

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

where a, b, m, n, r, s, p & q are all consts having any value, then the $I.F = x^h \cdot y^k$, where h & k are s.t. after multiplying I.F. the condition of exactness is satisfied.

Note:- h, k can be determined from the following two equations.

$$nh - mk = (m-n) + (mb - na)$$
$$qh - pk = (p-q) + (ps - qr)$$
provided $mq - np \neq 0$.

Eg ① Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$

Soln:- Here $M = xy - 2y^2$ & $N = -x^2 + 3xy$.

$\therefore \frac{\partial M}{\partial y} = x - 4y$ $\quad \frac{\partial N}{\partial x} = -2x + 3y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ Given D.E is not exact. But it is homogeneous.

Also

$\therefore x \cdot M + y \cdot N = x^2y - 2xy^2 - x^2y + 3xy^2 = xy^2 \neq 0$.

$\therefore$ By using Rule (I) :.

$$I.F = \frac{1}{x \cdot M + N \cdot y} = \frac{1}{xy^2}.$$

$\therefore$ Mutiply I.F to the given diffn eqn, $\therefore$ given eqn becomes,

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0 \text{ which is exact}$$

$\therefore$ Its G.S. is.

$$\Rightarrow \int_{y\text{-const}} M\,dx + \int \left(\text{Terms of } N \text{ not containing } x\right) dy = c.$$

$$\Rightarrow \int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\Rightarrow \boxed{\frac{x}{y} - 2\log x + 3\log y = c} \text{ is the G.S.}$$

(2) $y(1 + xy)\,dx + x(1 + xy + x^2y^2)\,dy = 0$

Soln:- Given eqn is not exact. To find I.F. using (rule 2), we have.

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{xy + x^2y^2 - xy - x^2y^2 - x^3y^3} = \frac{1}{-x^3y^3}$$

$\therefore$ Multiplying by $\frac{-1}{x^3y^3}$ to given equation

$$\therefore \left(\frac{-1}{x^3y^2} - \frac{1}{x^2y}\right)dx + \left(\frac{-1}{x^2y^3} - \frac{1}{xy^2} - \frac{1}{y}\right)dy = 0 \quad \ldots (1)$$

from eqn (1),

$$\therefore \frac{\partial M}{\partial y} = \frac{2}{x^3y^3} + \frac{1}{x^2y^2} = \frac{\partial N}{\partial x}.$$

$\therefore$ Eqn (1) is exact. $\therefore$ Its G.S. is given by

$$\frac{-1}{y^2}\int \frac{1}{x^3}dx - \frac{1}{y}\int \frac{1}{x^2}dx - \int \frac{1}{y}dy = c$$

$$\Rightarrow \frac{1}{2y^2x^2} + \frac{1}{xy} - \log y = c \quad \text{is the G.S.}$$

Que. (3) Solve. ~~(x sec² y − x²)~~ $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

Soln: Here $M = y^4+2y$, $N = xy^3+2y^4-4x$.

$$\therefore \frac{\partial M}{\partial y} = 4y^3+2, \quad \frac{\partial N}{\partial x} = y^3-4$$

$\therefore$ Given eqn is not exact

$\therefore$ By using rule(4), consider

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y(y^3+2)} = \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y}$$

$$\therefore I.F = e^{-3\int \frac{1}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$$

$\therefore$ Multiplying to given equation by $1/y^3$

$$\Rightarrow \left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0 \quad \text{--- (1)}$$

$$\Rightarrow M'dx + N'dy = 0$$

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$\therefore$ Equation (1) is exact & its G.S is.

$$\int_{y\ \text{const}} M'\,dx + \int \left(\text{Terms of } N \text{ not containing } x\right) dy = c$$

$\Rightarrow \left(y+\frac{2}{y^2}\right)\int dx + 2\int y\,dy = C$

$\Rightarrow \left(y+\frac{2}{y^2}\right)x + y^2 = C$ is the required General solⁿ.

(4) Solve:- $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$.

Solⁿ:- Here, $M = y^3 - 2x^2y$, $N = 2xy^2 - x^3$

& $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given equation can be written as,

$(y^3 dx + 2xy^2 dy) - (2x^2y\,dx + x^3 dy) = 0$

$\Rightarrow y^2(y\,dx + 2x\,dy) + x^2(-2y\,dx - x\,dy) = 0$.

using rule (5),

Here, $a=0$, $b=2$, $m=1$, $n=2$, $r=2$, $S=0$,

$p=-2$, $q=-1$.

$\therefore$ I.F = $\boxed{x^h . y^k}$

Multiplying I.F. to the given equation.

$\therefore$

$x^h y^k (y^3 - 2x^2y)dx + x^h y^k (2xy^2 - x^3)dy = 0$.

$\Rightarrow (x^h y^{k+3} - 2x^{h+2} y^{k+1})dx + (2x^{h+1} y^{k+2} - x^{h+3} y^k)dy = 0$.

$\frac{\partial M}{\partial y} = (k+3)x^h . y^{k+2} - 2(k+1)x^{h+2} . y^k$

$\frac{\partial N}{\partial x} = 2(h+1)x^h . y^{k+2} - (h+3)x^{h+2} . y^k$.

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But $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for exactness, which requires

$$k+3 = 2(h+1)$$
$$-2(k+1) = -(h+3) \quad \text{giving } h=1, k=1$$

$$\therefore I.F = x^h y^k = xy$$

Multiplying the given equation by xy, we get

$$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0 \quad \ldots (1)$$

Hence equation (1) is exact & its G.S. is

$$\int_{y\text{-const}} M'dx + \int \left[\text{Terms of } N \text{ not containing } x\right] dy = c$$

$$\Rightarrow y^4 \int x\,dx - 2y^2 \int x^3 dx = c$$

$$\Rightarrow y^4x^2 - y^2x^4 = c_1 \text{ is the required G.S.}$$

<u>Note</u>:- h, k can be determined by using short-cut method.

$$nh - mk = m - n + (mb - na)$$
$$\Rightarrow 2h - k = -1 + (2-0) \Rightarrow 2h - k = 1 \quad \ldots (1)$$

Also,

$$qh - pk = p - q + (ps - qr)$$
$$\Rightarrow -h + 2k = -1 + (0+2) \Rightarrow -h + 2k = 1 \quad \ldots (2)$$

solving (1) & (2), we get $h=1, k=1$

$$\therefore I.F = x^h y^k = xy.$$

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Type-(1) :- Linear Differential equation of the first order.

Defⁿ:- A differential equation is said to be linear if the degree of the diffⁿ eqⁿ is one.

Type (a) General form:- A D.E of the form

$$\boxed{\frac{dy}{dx} + Py = Q} \quad \text{---} (1)$$

where, P, Q are function of 'x' or const. is called linear differential equation of first order. in y.

Method of solution:-

An I.F. of eqⁿ (1) is $e^{\int P dx}$

∴ Therefore the G.S. is given by,

$$\boxed{y e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + C}$$

Type (b) General form:- lly,

$$\boxed{\frac{dx}{dy} + P.x = Q} \quad \text{---} (2)$$

where P, & Q are functions of 'y' or const. is also called linear diffⁿ eqⁿ. of the first order in x.

Method of solⁿ:- we write I.F of eqⁿ (2) as

$I.F = e^{\int P dy}$ and G.S. is given by.

$$\boxed{x \cdot e^{\int P dy} = \int Q e^{\int P dy} dy + C}$$

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Note:- The coefficient of $\frac{dy}{dx}$ or $\frac{dx}{dy}$ in linear diff^n equation must be equal to one.

Que① solve:- $(1+\sin y)\frac{dx}{dy} = 2y\cos y - x(\sec y + \tan y)$

Sol^n:- Divide by $(1+\sin y)$,

$$\frac{dx}{dy} + \left(\frac{\sec y + \tan y}{1+\sin y}\right)x = \frac{2y\cos y}{1+\sin y}.$$

$$\frac{dx}{dy} + \left(\frac{1+\sin y/\cos y}{1+\sin y}\right)x = \frac{2y\cos y}{1+\sin y}$$

$$\frac{dx}{dy} + (\sec y)x = \frac{2y\cos y}{1+\sin y} \quad \text{——— ①}$$

eq^n ① is linear in x with $P = \sec y$ &

$Q = \frac{2y\cos y}{1+\sin y}.$

$\therefore I.F = e^{\int \sec y\, dy}$
$= e^{\log(\sec y + \tan y)} = \sec y + \tan y$

G.S $\Rightarrow$ $x(\sec y + \tan y) = \cancel{1+\sin y}/\cancel{\cos y} \int Q.(\sec y + \tan y)\, dy + c$

G.S. $\Rightarrow x(\sec y + \tan y) = \int \left(\frac{2y\cos y}{1+\sin y}\right).\left(\frac{1+\sin y}{\cos y}\right)dy + c$

$$= \int 2y\, dy + c$$

$x(\sec y + \tan y) = y^2 + c$

$\therefore x(\sec y - \tan y) - y^2 = c.$ is the G.s.

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Que② solve:- $(1+y^2)+\left(x-e^{-\tan^{-1}y}\right)\dfrac{dy}{dx}=0$

Soln:- It can be written as,

$$(1+y^2)\frac{dx}{dy}+x=e^{-\tan^{-1}y}$$

$$\frac{dx}{dy}+\left(\frac{1}{1+y^2}\right).x=\frac{e^{-\tan^{-1}y}}{1+y^2}$$ which is linear in x.

$$\therefore I.F=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}.$$

$\therefore$ G.S is,

$$x\cdot e^{\tan^{-1}y}=\int\frac{e^{-\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy+c$$

$$=\int\frac{dy}{1+y^2}+c$$

$$\therefore x\cdot e^{\tan^{-1}y}=\tan^{-1}y+c$$ is the G.S.

③ solve $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1)$

Soln:- The given eqn can be written as,

$$\frac{dy}{dx}-\left(\frac{x-2}{x(x-1)}\right)y=\frac{x^3(2x-1)}{x(x-1)}$$

$$\therefore\frac{dy}{dx}-\left(\frac{x-2}{x(x-1)}\right)y=\frac{x^2(2x-1)}{(x-1)}$$

Here, $P=-\dfrac{(x-2)}{x(x-1)}$ & $Q=\dfrac{x^2(2x-1)}{x-1}$

$$\therefore I.F=e^{\int P\,dx}=e^{\int\frac{-(x-2)}{x(x-1)}dx}=e^{\int\left(\frac{-2}{x}+\frac{1}{x-1}\right)dx}$$

$$= e^{-2\log x + \log(x-1)}$$

$$I.F = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, the general soln is,

$$y.(I.F) = \int Q(I.F)\,dx + c$$

$$\Rightarrow y\left(\frac{x-1}{x^2}\right) = \int \frac{x^2(2x-1)}{x-1} \cdot \left(\frac{x-1}{x^2}\right) dx + c$$

$$= \int (2x-1)\,dx + c$$

$$\Rightarrow \frac{y(x-1)}{x^2} = x^2 - x + c$$ is the G.S.

Type (8) Equation Reducible to Linear form. (Bernoulli's equation):-

*(1) Bernoulli's Differential eqn:-

A differential eqn of the form

$$\boxed{\frac{dy}{dx} + Py = Q.y^n}$$ is called as Bernoulli's differential equation.

Method of soln:- we divide by y^n.

$$\therefore y^{-n}\frac{dy}{dx} + P.y^{1-n} = Q \quad \cdots (1)$$

put $y^{1-n} = u \Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$

$$(1) \Rightarrow \frac{1}{1-n}\frac{du}{dx} + P.u = Q$$

or $$\frac{du}{dx} + (1-n)P.u = (1-n)Q.$$

which is linear & therefore can be solved by using linear diffⁿ eqⁿ method.

* (2) The differential eqⁿ of the form

$\boxed{\frac{dx}{dy} + P.x = Q.x^n}$ is also called as Bernoulli's diffⁿ eqⁿ.

Now, dividing by x^n.

$$\Rightarrow x^{-n}\frac{dx}{dy} + Px^{1-n} = Q \quad \text{——— (2)}$$

put $x^{1-n} = u \quad \Rightarrow (1-n)x^{-n}\frac{dx}{dy} = \frac{du}{dy}.$

$$\therefore x^{-n}\frac{dx}{dy} = \frac{1}{(1-n)}\frac{du}{dy}.$$

put in eqⁿ (2), we get.

$$\frac{1}{(1-n)}\frac{du}{dy} + P.u = Q$$

$\Rightarrow \frac{du}{dy} + (1-n)P.u = (1-n)Q$ which is linear. & ∴ can be solved.

* (3) Equation of the form:-

$\boxed{f'(y)\frac{dy}{dx} + Pf(y) = Q}$ are at once reducible to the linear form by the substitution of $f(y) = u$ &

$f'(y)\frac{dy}{dx} = \frac{du}{dx}$ transforming the eqⁿ

to $\frac{du}{dx} + P.u = Q$ which is linear.

(4) similarly for, $\boxed{f'(x)\frac{dx}{dy} + Pf(x) = Q}$,

we substitute $f(x) = u$ & $f'(x)\frac{dx}{dy} = \frac{du}{dy}$.

Equation reduces to,

$\frac{du}{dy} + P.u = Q$ which is linear.

Illustrations on Equations Reducible to linear form:-

EX (1):- solve $\cos y - x\sin y\frac{dy}{dx} = \sec^2 x$ — (1)

Soln:- put $\cos y = u$

$\Rightarrow -\sin y\frac{dy}{dx} = \frac{du}{dx}$

$\therefore$ eqn (1) becomes

$\Rightarrow u + x\frac{du}{dx} = \sec^2 x$.

$\therefore \frac{du}{dx} + \frac{1}{x}.u = \frac{\sec^2 x}{x}$ ... which is linear in u

Here,

$P = 1/x$, $Q = \sec^2 x/x$

$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

G.S $\Rightarrow$ $u.(I.F) = \int Q.(I.F)\,dx + c$

$u.x = \int \frac{\sec^2 x}{x}\,x\cdot dx + c$.

$u.x = \int \sec^2 x \cdot dx + c$

$\therefore$ $\boxed{x.\cos y = \tan x + c}$ is the G.S.

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EX.(2) solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Soln:- Given diffn eqn can be written as

$$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$$

This is Bernoulli's equation, Divide by y^3

$$+y^{-3}\frac{dy}{dx} - x.y^{-2} = -e^{-x^2} \quad \text{——— (1)}$$

put $y^{-2} = u$

$$\Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{du}{dx}$$

eqn (1) $\Rightarrow$ $$-\frac{1}{2}\frac{du}{dx} - xu = -e^{-x^2}$$

$$\therefore \frac{du}{dx} + (2x)u = 2e^{-x^2}$$

Here, $P = 2x$, $Q = 2e^{-x^2}$

$$\therefore I.F = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ General soln $\Rightarrow$ $u.(I.F) = \int Q.(I.F)\,dx + c$

$$\therefore u.e^{x^2} = \int 2e^{-x^2}.e^{x^2}\,dx + c$$

$$\therefore ue^{x^2} = \int 2\,dx + c$$

$\therefore \frac{e^{x^2}}{y^2} = 2x + c$ is the required G.S.

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EX:- (3) Solve $\cos x \frac{dy}{dx} + y\sin x = \sqrt{y\sec x}$

Sol^n:- Given $\frac{dy}{dx} + y\cdot\frac{\sin x}{\cos x} = \sqrt{y}\sec^{3/2} x$

This is Bernoullis diffth eq^n. Dividing by $\sqrt{y}$

$$\therefore y^{-1/2}\frac{dy}{dx} + y^{1/2}\tan x = \sec^{3/2} x. \quad \text{—} (1)$$

put $y^{1/2} = u$

$$\therefore \frac{1}{2} y^{-1/2}\frac{dy}{dx} = \frac{du}{dx}.$$

$$\Rightarrow y^{-1/2}\frac{dy}{dx} = 2\frac{du}{dx}$$

$\therefore$ eq^n (1) becomes,

$$2\frac{du}{dx} + u\cdot\tan x = \sec^{3/2} x.$$

$$\Rightarrow \frac{du}{dx} + \left(\frac{1}{2}\tan x\right)u = \frac{1}{2}\sec^{3/2} x.$$

$\therefore P = \frac{1}{2}\tan x$, $Q = \sec^{3/2} x \cdot \frac{1}{2}$

$$\therefore I.F = e^{\int P dx} = e^{\frac{1}{2}\int \tan x\, dx}.$$

$$= e^{\frac{1}{2}\log\sec x} = e^{\log\sqrt{\sec x}}$$

$$\therefore I.F = \sqrt{\sec x}.$$

$\therefore$ G.S $\Rightarrow$

$$u(I.F) = \int Q.(I.F)\, dx + c$$

$$u\sqrt{\sec x} = \int \frac{1}{2}\sec^{3/2} x.\sqrt{\sec x}\, dx + c.$$

$$u\sqrt{\sec x} = \frac{1}{2}\int \sec^2 x\, dx + c$$

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$$y\sqrt{\sec x} = \frac{1}{2}\tan x + c$$

$\Rightarrow \sqrt{y\sec x} = \frac{1}{2}\tan x + c$ is the required G.s.

Transformation to polars:-

The eqⁿ transformed to polars by using $x = r\cos\theta$, $y = r\sin\theta$; it may be convenient to solve the diffⁿ eqⁿ in the new variables r and θ.

In such cases, we note that, since

$x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$, $\theta = \tan^{-1} y/x$

$\therefore 2x + 2y = 2r\,dr$

$\Rightarrow \underline{x\,dx + y\,dy = r\,dr}$

Also, $x\,dy - y\,dx = r\cos\theta(r\cos\theta\,d\theta + \sin\theta\,dr) - r\sin\theta(\cos\theta\,dr - r\sin\theta\,d\theta)$

$\therefore \boxed{x\,dy - y\,dx = r^2 d\theta}$

Que① solve:- $x^2(x\,dx + y\,dy) + y(x\,dy - y\,dx) = 0$

Solⁿ:-

Transforming to polar co-ordinates by using $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore x^2+y^2 = r^2$.

$x\,dx + y\,dy = r\,dr$.

Also,

$$\frac{y}{x} = \tan\theta.$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta.$$

we have, $x\,dx + y\,dy + \frac{y(x\,dy - y\,dx)}{x^2} = 0$

$\Rightarrow r\,dr + r\sin\theta\cdot\sec^2\theta\,d\theta$

$\Rightarrow dr + \tan\theta\cdot\sec\theta\cdot d\theta = 0$

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$$\int dr + \int \sec\theta \tan\theta \, d\theta = c$$

$\Rightarrow r + \sec\theta = c$

Hence $\sqrt{x^2+y^2} + \dfrac{\sqrt{x^2+y^2}}{x} = c$ $\left(\because \cos\theta = \dfrac{x}{r}\right.$

$\therefore \sec\theta = \dfrac{r}{x}$

$\Rightarrow \sqrt{x^2+y^2}\,(x+1) = cx$ is the G.S.

Equation both are Homogeneous & Exact:-

Suppose that M & N are Homogeneous funct^n of x & y of degree n $(n \neq -1)$ further, suppose that $M dx + N dy = 0$ is Exact. Then the general sol^n is

$Mx + Ny = c$

e.g. 1 $(x^3 + 3y^2x)\,dx + (y^3 + 3x^2y)\,dy = 0.$

$\Rightarrow$ Here given equation is Homo. & Exact.

$\therefore$ The G.S. is. $Mx + Ny = c$

$$x(x^3 + 3y^2x) + y(y^3 + 3x^2y) = c$$

$\therefore \Rightarrow x^4 + 6x^2y^2 + y^4 = c$

—||—— OX ——||—